Mathematics 222B Lecture 10 Notes

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1 L^2-Based Elliptic Regularity

1.1 Regularity theory for the Poisson equation

Last time, we discussed solvability for elliptic PDEs. Now we will talk about the regularity of solutions to elliptic PDEs. Here is a prototypical example.

Example 1.1. Consider the Poisson equation $-\Delta u = f$ in U, where $f \in H^k(U)$ or $C^{k,\alpha} = \{u \in C^k(U) : \partial^\alpha u \in C^{0,\alpha}(U)\, \forall |\alpha| = k\}$. The idea is that u should be more regular than f by order 2. Interior regularity says that for all $V \subseteq\subseteq U$ (notation meaning V is bounded and $\overline{V} \subseteq U$),

$$\|u\|_{H^{k+2}(V)} \leq C\|f\|_{H^k(V)} + C\|u\|_{L^2(U)}.$$

Similarly,

$$\|u\|_{C^{k+2,\alpha}(V)} \leq C\|f\|_{C^{k,\alpha}(V)} + C\|u\|_{L^\infty(U)}.$$

In general, the constant C can depend on the domain V.

The first of these statements is referred to as L^2**-based regularity theory**, and the second is referred to as **Schauder theory**. We will think about L^2-based regularity theory for now and discuss Schauder theory later.

For L^2-based regularity theory, the key idea is integration by parts (the energy method).[1] We will make a simplifying that $u \in H^{k+2}(V)$; this is not assuming everything because from this qualitative fact, we will derive a quantitative bound. This assumption allows us to commute the equation with derivatives. We have not said any assumptions about the boundary, which may seem like an issue with integration by parts, but this is why we are discussing *interior* regularity. We will solve this with a cutoff function.

[1] Fraydoun Rezakhanlou says that he is an analyst, a PDE-ist, and a probabilist. He is an analyst because he uses the Cauchy-Schwarz inequality, a probabilist because he uses Chebyshev's inequality, and a PDE-ist because he uses integration by parts.

Let ζ be a nonnegative, smooth cutoff function which equals 1 in V and equals 0 near ∂U. Then (squaring ζ in anticipation of a nice L^2 trick),

$$\begin{aligned}\int_U fu\zeta^2\,dx &= \int_U -\Delta u u\zeta^2 \\ &= \sum_{j=1}^d \int_U \underbrace{\partial_j u \partial_j(u\zeta^2)}_{\partial_j u\zeta^2 + 2u\zeta\partial_j\zeta}\,dx\end{aligned}$$

Note that we have no boundary term in the integration by parts thanks to ζ.

$$= \sum_{j=1}^d \int (\partial_j u)^2\zeta^2 + 2\partial_j u u\zeta\partial_j\zeta\,dx.$$

Rearrange this to get

$$\int_U |Du|^2\zeta^2\,dx \leq \left|\int_U fu\zeta^2\,dx\right| + \underbrace{2\left|\int_U u\zeta Du\cdot D\zeta\,dx\right|}_{\leq 2(\int_U |Du|^2\zeta^2)^{1/2}(\int_U u^2|D\zeta|^2\,dx)^{1/2}}$$

To control this right term, we use the AM-GM inequality $ab \leq \frac{a}{2}+\frac{b}{2}$. But we can weight this by $\sqrt{\varepsilon}$ on a and $\frac{1}{\sqrt{\varepsilon}}$ on b to get the inequality $ab \leq \varepsilon\frac{a^2}{2} + \frac{1}{\varepsilon}\frac{b^2}{2}$. This bounds

$$2\left(\int_U |Du|^2\zeta^2\right)^{1/2}\left(\int_U u^2|D\zeta|^2\,dx\right)^{1/2} \leq \varepsilon\int_U |Du|^2\zeta^2\,dx + \frac{1}{\varepsilon}\int_U u^2|D\zeta|^2\,dx.$$

Now set $\varepsilon = 1/2$ to absorb the first term to the right hand side.

This gives

$$\begin{aligned}\frac{1}{2}\int_U |Du|^2\zeta^2\,dx &\leq \left|\int_U fu\zeta^2\right| + 2\int_U u^2|D\zeta|^2\,dx \\ &\leq \|f\|_{L^2(U)} + \|u\|_{L^2(U)},\end{aligned}$$

and we lower bound the left hand side by $\frac{1}{2}\int_V |Du|^2\,dx$. For the actual result, we could have upgraded the $\|f\|_{L^2(U)}$ to $\|f\|_{H^1(U)}$ by using an additional cutoff argument.

What about higher regularity? Suppose $k+2=2$. Then if $-\Delta u = f$, we get

$$-\Delta\partial_j u = \partial_j f,$$

where $\partial_j u \in H^1$, so we can do integration by parts. Now apply the case $k=1$ to get

$$\int_V |D\partial_j u|^2\,dx \leq \left|\int_U \partial_j f\partial_j u\zeta^2\,dx\right| + \|\partial_j u\|_{L^2(U)}$$

Bound the first term by (using the same AM-GM trick)

$$\left|\int_U f\partial_j^2 u\zeta^2\,dx\right| \le \frac{1}{4\varepsilon}\int_U f^2\zeta^2\,dx + \varepsilon\int_U |\partial_j u|^2\zeta^2\,dx.$$

Absorb the second term to the right hand side to get

$$\int_U |D\partial_j u|^2\zeta^2\,dx \le C\int_U f^2\,dx + C\|Du\|_{L^2(U)}^2.$$

We want to change the last term into $\|u\|_{L^2(U)}$. Our tool to do this is the H^1 bound we just proved. But this needs us to have a domain in the interior of U. However, note that if we define $V \subseteq\subseteq W \subseteq\subseteq U$, we can replace this term on the the right hand side by $C\|Du\|_{L^2(W)}$. Then we use the H^1 bound $\|Du\|_{L^2(W)} \le \|f\|_{L^2(U)} + \|u\|_{L^2(U)}$. In conclusion, we get

$$\|D\partial_j u\|_{L^2(V)} \le C\|f\|_{L^2(U)} + C\|u\|_{L^2(U)}$$

for all j. Combined with the H^1 bound, this gives the H^2 bound

$$\|u\|_{H^2(V)} \le C\|f\|_{L^2(U)} + C\|u\|_{L^2(U)}.$$

1.2 L^2-regularity for elliptic operators

For the full L^2-regularity theorem, we have an elliptic operator

$$Pu = -\partial_j(a^{j,k}\partial_k u) + b^j\partial_j u + cu,$$

where $u : U \to \mathbb{R}$ and U is an open subset of $\mathbb{R}^d$. We also assume $a(x) \succ \lambda I$ for some $\lambda > 0$ for all $x \in U$. Also assume $a, b, c \in L^\infty(U)$ (although the natural assumption for $d \ge 3$ is actually $a \in L^\infty, b \in L^d, c \in L^{d/2}$). For the H^2 bound, we also make the assumption that $\partial a \in L^\infty(U)$; this comes from the fact that if we want to commute the derivative as in the argument above, we must be able to deal with the derivative of the coefficients $a^{i,j}$.

Theorem 1.1 (H^2 elliptic regularity)**.** *Let $u \in H^1(U)$ be a weak solution to $Pu = f$ on U, and let $f \in L^2(U)$. Then for all $V \subseteq\subseteq U$, $u \in H^2(V)$, and*

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

The proof of this theorem is the same as the previous argument but with some minor adjustments. The main step is integration by parts. Formally,

$$\begin{aligned}\int_U -\partial_j(a^{j,k}\partial_k v)v\zeta^2\,dx &= \int_U a^{j,k}\partial_k v\partial_j v\zeta^2\,dx + \int_U a^{j,k}\partial_k v v\zeta\partial_j\zeta\,dx \\ &\ge \lambda\int_U |Du|^2\zeta^2\,dx - \|a\|_{L^\infty}\cdot\underbrace{\int |Du|\zeta|v||D\zeta|\,dx}_{\le \frac{\lambda}{2}\frac{1}{\|a\|_{L^\infty}}|Dv|^2\zeta^2 + \frac{1}{\lambda}\|a\|_{L^\infty}|v|^2|D\zeta|^2}\end{aligned}$$

$$\geq \frac{\lambda}{2}\int_U |Dv|^2\zeta^2\,dx - \frac{\|a\|_{L^\infty}^2}{\lambda}\int_U |v|^2|D\zeta|^2\,dx.$$

Since we do not know a priori that $u \in H^2(V)$, need to modify the proof idea to commute the equation with difference quotients instead of derivatives.

Definition 1.1. If $k \in \{1,\ldots,d\}$ and $h \in \mathbb{R}\setminus\{0\}$, the **difference quotient** is

$$D_k^h v(x) = \frac{v(x+hek)-v(x)}{h}.$$

This converges to $\partial_k v(x)$ as $h \to 0$.

Proof. Step 0: Note that for $u \in H^1(U)$,

$$Pu = f \text{ in } U \iff \langle Pu,\varphi\rangle = \langle f,\varphi\rangle \qquad \forall \varphi \in C_c^\infty(U)$$

Here, $Pu \in H^{-1}(U)$, $f \in L^2 \subseteq H^{-1}$.

$$\iff \langle Pu\varphi\rangle = \langle f,\varphi\rangle \qquad \forall\varphi \in H_0^1(U) \quad (= (H^{-1}(U))^*)$$

When we did our a priori estimate last time, we used approximation of u by smooth functions. However, here, we want to show that we have extra regularity, so the equivalent of approximation is this step above.

$$\iff \int_U a^{j,k}\partial_j u\partial_k\varphi + b^j\partial_j u\varphi + cu\varphi\,dx = \int_U f\varphi\,dc \qquad \forall\varphi\in H_0^1(U).$$

Step 1: Now commute the equation with D_j^h. Note that the Leibniz rule holds:

$$D_h^h(uv)(x) = D_j^h u(x)v(x) + u(x+h)D_j^h v(x).$$

This comes from

$$uv(x+h) - uv(x) = (u(x+h)-u(x))v(x) + \underbrace{u(x+h)}_{=:u^h(x)}(v(x+h)-v(x)).$$

Now

$$\begin{aligned}D_j^h f &= D_j^h(-\partial_j a^{j,k}\partial_k u + b^j\partial_j u + cu)\\ &= -\partial_\ell(a^h)^{j,k}\partial_k D_j^h u + (b^h)^j\partial_\ell D_j^h u + c^h D_j^h u - \partial_\ell(D^h a)^{\ell,k}\partial_k u + (D_j^h b)^\ell\partial_\ell u + D_j^h cu.\end{aligned}$$

Rearrange this as

$$-\partial_\ell((a^h)^{\ell,k}\partial_k D_j^h u) = \widetilde{f}_1^h,$$

where f_1^h is everything else. Now

$$\langle -\partial_\ell(a^h)^{\ell,k}\partial_k D_j^h u,\varphi\rangle = \langle\widetilde{f}_1^h,\varphi\rangle \qquad \forall\varphi\in H_0^1(U),$$

where the left hand side equals

$$\int (a^h)^{\ell,k} \partial_k (D_j^h) u \partial_\ell \varphi \, dx.$$

Step 2: "$\varphi = \partial_j u \zeta^2$": Choose $\varphi = D_j^h \zeta^2 \in H_0^1(U)$. By the integration by parts idea, we get

$$\frac{\lambda}{2} \int_U |DD_j^h u|^2 \zeta^2 \; dx \leq \cdots \widetilde{f}_1 D_j^h u.$$

One treats the right hand side like before, treating $D_j^h u$ like $\partial_j u$. To make this precise, we need the following lemma:

Lemma 1.1 (from Ch 5 in Evans)**.** *Let $V \subseteq\subseteq U$.*

1. *If $u \in W^{1,p}$, $\|D_j^h u\|_{L^p(V)} \leq C \|\partial_j u\|_{L^p(U)}$ for $|h| \ll 1$.*
2. *Assume $u \in L^p$. For $h \ll 1$, if $\|D_j^h u\_L^p(V) \leq A$, then $\partial_j \in L^p$, and $\|\partial_j u\|_{L^p(V)} \leq A$.*

This finishes off the proof. $\square$